Design of Helicopter Stabilization Systems Using Optimal Control Theory

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A technique is described for the design of multivariable controllers for helicopters, based on results obtained recently in optimal control theory. For a specified quadratic performance index, the optimal feedback controller is obtained by solving a matrix Riccati equation. A method of determining the performance index which yields a desired response is outlined in the paper. The solution represents the fastest response that can be obtained for a specified amplitude of anticipated disturbance and a limited controller output. An example is worked out in detail to indicate the effectiveness of the method.

Nomenclature

A = $n \times n$ system matrix
a_0, a_1 = constant coefficients of the differential equation describing the model
B = $n \times r$ input matrix
C = $m \times m$ model matrix
e = n-dimensional vector; difference between model and aircraft state outputs
F, Q, R = $n \times n$, $n \times n$, $r \times r$ symmetric matrices
H = Hamiltonian function
J = performance index
K = $n \times n$ symmetric matrix; solution to the Riccati equation
K_1 = $n \times n$ feedback gain matrix
K_2 = $n \times m$ feedforward gain matrix
L = $n \times m$ matrix used to determine which aircraft states variables are to follow model state variables
m = order of the model
n = order of the system
y = state variable output of model
$\dot{y}(0)$ = initial condition on the model rate
r = order of the control vector
T = terminal time
t = time variable
t_0 = initial time
t_{10} = minimum time for the longitudinal velocity to reach zero in response to a longitudinal disturbance
u = r-dimensional control vector
u_C = main rotor collective pitch control input; change from trim
u_{OPT} = r-dimensional optimal control vector
u_P = longitudinal cyclic pitch control input; change from trim
u_R = lateral cyclic pitch control input; change from trim
u_T = tail rotor collective pitch control input; change from trim
x = n-dimensional state vector of the aircraft
x_0 = initial conditions on the system state vector
x_{10} = maximum amplitude of a disturbance in longitudinal velocity to which the system can respond without exceeding the limit on the controller output
z = $(n + m)$-dimensional state vector of aircraft and model
θ_F, ϕ, ψ = pitch, roll, and yaw angle; change from trim
λ = n-dimensional vector of Lagrange multipliers
μ_X, μ_Y, μ_Z = normalized longitudinal, lateral, and vertical velocity; change from trim
τ = a dummy time variable

Subscripts and superscripts

i = ith element of a vector
ii = iith element of a matrix
ss = steady state
$[\ ]^{-1}$ = inverse of a matrix
$[\ ]'$ = transpose of a matrix
$(\dot{\ })$ = derivative with respect to time

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1.0 Introduction

MISSION requirements for helicopters dictate that a large amount of total flight time be spent at hover. Most of the tasks performed while hovering require the aircraft to remain fixed with respect to the ground in the presence of disturbances. This, in general, is almost impossible for a pilot flying a helicopter without stability augmentation since the transfer function relating pilot control motion to aircraft translational displacement contains four integrations.[1] It is well known that the pilot's effort can be greatly reduced by introducing feedback signals into the flight control channels. The determination of these feedback signals represents the control problem discussed here.

Stability augmentation systems for helicopters are normally designed using frequency domain—transfer function techniques. The usual method of attack is to separate the aircraft's equations of motion into two uncoupled modes, a longitudinal and a lateral mode. The equations for the longitudinal mode relate aircraft pitch, longitudinal, and vertical motion to the collective and longitudinal cyclic controls. The equations for the lateral mode relate aircraft roll, yaw, and lateral motion to the pedals and the lateral cyclic controls. Using such an approach, the compensation for each mode is designed so as to give a satisfactory transient response using root locus, Bode, and Nyquist techniques. A considerable amount of trial and error, and experience, is involved in this procedure because the compensation chosen in one channel affects the choice of compensation in the others. Multivariable feedback into a channel is avoided since this generally makes the problem intractable. The difficulties associated with the foregoing approach are readily apparent. Cross-coupling effects between the longitudinal and lateral modes, which can be quite significant, are neglected. In addition, the feedback parameters are decided a priori; cross-feedback terms, which could be beneficial, are not normally considered.

The technique described in this paper is a systematic and efficient method for designing multivariable controllers for helicopters based upon well-known results in optimal control theory. This method yields a multiloop system in which a linear combination of all aircraft states is fed back into each control channel. Consequently, every feasible feedback path is considered.

2.0 Optimal Control of Linear Systems

The problem of determining an optimal controller for a linear system to minimize a quadratic criterion may be stated as follows: given the multi-input, multi-output dynamical system represented by a set of linear first-order differential equations of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \qquad x(t_0) = x_0 \tag{1}$$

determine the control function $u(t)$, specified over the interval $t_0 \leq t \leq T$ (where t_0, T are specified), which minimizes the performance index J;

$$J = \frac{x(T)'Fx(T)}{2} + \frac{1}{2}\int_{t_0}^{T} [x(t)'Qx(t) + u(t)'Ru(t)]dt \tag{2}$$

where

$x(t)$ = vector of n state variables (roll angle, pitch rate, etc.)
$u(t)$ = a vector of r forcing functions (e.g., pedals and collective stick inputs)
A, B = $n \times n$ and $n \times r$ matrices
Q, F, R = $n \times n$, $n \times n$, and $r \times r$ symmetric matrices, respectively

This problem has been considered in great detail by many authors,[2,3] and is the only problem known at present for which a general solution can be derived. For the benefit of the uninitiated reader, the method of determining $u(t)$ is outlined below.

It can be shown using the classical calculus of variations,[4] or Pontryagin's Maximum Principle,[5] that the optimal control function u_{OPT} minimizes the Hamiltonian H;

$$H = \tfrac{1}{2}(x'Qx + u'Ru) + \lambda(Ax + Bu) \tag{3}$$

where λ is a vector of Lagrange multipliers. Since $u(t)$ is unconstrained

$$\partial H/\partial u = Ru_{\text{OPT}} + B'\lambda = 0$$
$$u_{\text{OPT}} = -R^{-1}B'\lambda \tag{4}$$

The necessary conditions for optimization of the performance index are given by the canonical equations

$$\dot{x} = \partial H/\partial\lambda = Ax + Bu = Ax - BR^{-1}B'\lambda$$
$$\dot{\lambda} = -(\partial H/\partial x) = -Qx - A'\lambda \tag{5}$$
$$x(t_0) = x_0 \qquad \lambda(T) = Fx(T)$$

These equations (5) may be solved for the auxiliary variable λ in a straightforward manner. The solution indicates that λ is a linear function of x. Hence

$$u_{\text{OPT}} = -R^{-1}B'Kx \tag{6}$$

K is an $n \times n$ matrix which is the solution of a matrix Riccati equation

$$\dot{K} = KBR^{-1}B'K - KA - A'K - Q \tag{7}$$

with the boundary condition $K(T) = F$.

The matrix Riccati equation together with Eq. (6) constitutes the total solution to the regulator problem. Equation (6) gives the optimal feedback as a linear combination of all aircraft states into each control channel or a total of $r \times n$ feedback terms. Thus, every possible feedback path is considered.

Kalman[6] has shown that this technique must yield a stable closed-loop system when the R matrix is positive definite and the Q matrix is positive semidefinite. It should be noted, however, that these conditions are sufficient, not necessary. If the regulation period T in Eq. (2) is some finite value, the regulator gains will, in general, vary with time. If, however, the regulation period approaches infinity, and the system dynamics and performance index are time invariant (A, B, Q, R are constant matrices), the regulator gains will approach constants. It is not possible to specify any particular regulation period for the problem being considered since disturbances in the helicopter will be random in nature. Thus, the choice of an infinite regulation period, and the corresponding use of only the steady-state value of K, is the only reasonable choice that can be made.

In practice, solution of the Riccati equation is facilitated by making a change in the time variable ($\tau = T - t$), so that the final condition on K in forward time becomes an initial condition on K in backward time. Only $n(n + 1)/2$ equations have to be solved simultaneously since K is a symmetric matrix. The equation can be solved on a digital computer using a numerical integration procedure, given A, B, Q, R, F, t_0, and T.

3.0 Models

Thus far, only the problem of system regulation about $x \equiv 0$ has been treated. In the present design study, it is necessary for the helicopter to follow pilot input commands (the servo problem) as well as regulating in the presence of disturbances.

When the response of the aircraft is to be optimized for a set of pilot inputs, two distinct approaches can be used. The first approach uses the given pilot input to generate a control input to the system as the solution of an adjoint equation. The second approach uses a model such that the response of the model with different initial conditions approximates the pilot's input. By adjoining the model to the given system, the problem of tracking the pilot's command signals can be converted into a regulator problem. It is this second method which is used in this paper.

Problem: given a system described by the vector differential equation $\dot{x} = Ax + Bu$ and a model $\dot{y} = Cy$ where x, u, and y are $n \times 1$, $r \times 1$, and $m \times 1$ vectors, and A, B, and C are $n \times n$, $n \times r$, and $m \times m$ matrices, determine the control function $u(t)$ which minimizes the performance index

$$J = \tfrac{1}{2}\int_{t_0}^{T} (e'Qe + u'Ru)dt \tag{8}$$

where $e = x - Ly$ and L is an $n \times m$ matrix.

Adjoining the model to the system, we have the set of $n + m$ equations

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} [u] \tag{9}$$

or

$$\dot{z} = A_1 z + B_1 u \tag{10}$$

where

$$A_1 \equiv \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} \qquad B_1 \equiv \begin{bmatrix} B \\ 0 \end{bmatrix}$$

The performance index (8) may be expressed as

$$J = \tfrac{1}{2} \int_{t_0}^{T} (z'Q_1z + u'Ru)dt \qquad (11)$$

where

$$Q_1 = \begin{bmatrix} Q & -QL \\ -L'Q & L'QL \end{bmatrix}$$

Equations (10) and (11) are now of the same form as Eqs. (1) and (2) and the entire system can be solved using the technique of Sec. 2.0. The solution of this problem yields two gain matrices K_1 and K_2. The $n \times n$ feedback gain matrix K_1 is identical to that obtained for the regulator case, but the $n \times m$ feedforward gain matrix K_2 depends upon the model as shown below.

Using the results of the previous section, the control function may be expressed as

$$\begin{aligned} u(t) &= -R^{-1}B_1'Kz \\ &= -R^{-1}B'K_1x - R^{-1}B'K_2y \end{aligned} \qquad (12)$$

where

$$K = \begin{bmatrix} K_1 & K_2 \\ K_2' & K_3 \end{bmatrix} \qquad \text{an } (m+n) \times (m+n) \text{ matrix}$$

and K satisfies the matrix Riccati equation

$$\dot{K} = KB_1R^{-1}B_1'K - KA_1 - A_1'K - Q_1 \qquad K(T) = 0 \qquad (13)$$

From Eq. (13), it is seen that K_1 is identical to that obtained in the previous case. The feedforward gain matrix K_2 satisfies the equation

$$\dot{K}_2 = K_1BR^{-1}B'K_2 - K_2C - AK_2' - QL \qquad (14)$$

and depends on K_1 and the model C.

It should be noted that for every class of inputs (e.g., pulse having a specified width), a model has to be constructed and the corresponding feedforward gains determined. When different types of pilot inputs are anticipated, the different gain matrices corresponding to these inputs are determined and a suitable compromise made on the basis of the frequency of these inputs.

4.0 Performance Index

As described in Sec. 2.0, the solution of the optimal regulator problem is well defined. Although the equations for a complex multi-input, multi-output plant may be somewhat unwieldy, the solution is quite straightforward. The principal difficulty lies not in the solution, but in the choice of a suitable performance index. The solution is optimal in the sense that the chosen performance index is minimized, but different optimal solutions can be obtained by altering the Q and R matrices. The performance of an aircraft control system is ultimately judged by a pilot using a subjective criterion. The designer must rely upon his experience and judgment to indicate the nature of the response curves that the pilot will find satisfactory. Since only a tenuous relationship exists between the performance index and the desired performance, a certain amount of trial and error cannot be avoided.

The performance index may be interpreted as a quantitative measure of the system performance. The R matrix penalizes the control input required. The Q matrix penalizes the error in maintaining a desired trajectory. Thus, a large value of the ratio Q/R (ratio of elements in Q to corresponding elements in R) requires the system to follow the desired trajectory with little regard to the input control magnitude; conversely, a small ratio allows a large trajectory error but restricts the magnitude of the input control. The F matrix penalizes the error between the final state and the desired terminal state.

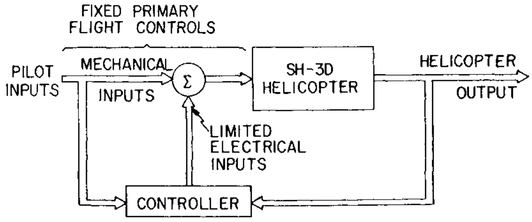


Fig. 1 SH-3D stabilization system.

The usual design procedure is to choose the Q and R matrices, obtain an optimal solution, and then update Q and R on the basis of the transient response. Normally the Q and R matrices are chosen to be diagonal for convenience in the updating procedure. If the solution requires too large an amplitude of u_i, the element R_{ii} is increased; if the x_i state possesses a large transient about its desired value (zero for the normal regulator problem), the element Q_{ii} is increased. This procedure may be viewed as a gradient method in the space of the elements of the Q and R matrices. The technique outlined in Sec. 2.0 is merely a convenient tool for obtaining the optimal solution at every step.

In this paper, an extension of the foregoing method is used to determine a set of curves (Sec. 5.0) relating disturbance amplitude, system speed of response, and control effort to the parameters in the performance index (i.e., elements of Q and R). Once these curves are obtained for a specific helicopter, the gains which yield the desired response are easily determined.

One may question why Q and R are updated instead of K. The answer lies in the updating procedure. Given a particular optimal solution, it is not clear how K should be adjusted to generate the desired transient response, and there is no systematic procedure to achieve this response. However, the qualitative relationship between Q and R and the state and control variables is sufficiently clear to allow systematic and efficient adjustment of Q and R.

5.0 Hover Stability Augmentation System Design

The helicopter selected here for the hover design study is a Sikorsky SH-3D Sea King. The prime mission of this aircraft is to detect submerged submarines through the use of a sonar ball which is lowered into the water while the aircraft is hovering at an altitude of approximately 40 ft. Hence, the aircraft spends the majority of its flight time hovering over a fixed point on the water.

The problem of maintaining the hovering helicopter about a fixed point, in the presence of disturbances, is precisely the regulator problem of Sec. 2.0. It is further required that the aircraft ground speed and yaw rate follow pilot input commands with a high degree of fidelity. This problem lends itself to the use of the model following concept described in Sec. 3.0.

Figure 1 indicates the configuration of the over-all system. The direct feedforward path between the pilot and the aircraft represents the pilot's primary flight controls. These mechanical linkages cannot be altered in the design. Controller output excursions are limited, and the parameters in the controller must be time invariant. The design problem is to determine the fixed parameters of the controller to

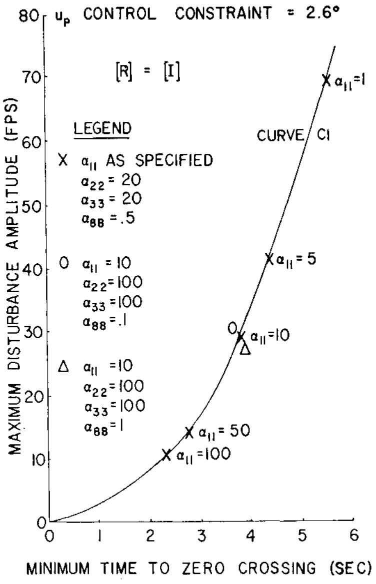


Fig. 2 Maximum disturbance amplitude vs minimum longitudinal velocity response time with 10% controller constraint.

achieve "satisfactory" regulation and control in the presence of the above constraints.

For the regulator problem, the pilot inputs are identically zero, and the entire input is determined by the controller. However, while following a desired output, the control input consists of the pilot input and the controller output. The amplitude of the controller output is limited to $\pm 10\%$ of the total available range of the mechanical inputs. This limit enables the pilot to successfully recover from a hardover failure in the augmentation system. The problem is to determine the controller parameters (subject to the limiter constraint) to obtain satisfactory response for regulation as well as for a specified class of pilot inputs.

Equations of Motion

The hover dynamics of the SH-3D helicopter, for small deviations from trim conditions, may be represented by the following linear differential equations of motion:

$$\dot{x}_1 = -0.016x_1 - 0.0047x_2 - 0.0001x_5 - 0.05x_6 + 0.0025x_7 + 0.05u_P + 0.005u_C$$

$$\dot{x}_2 = 0.0047x_1 - 0.033x_2 - 0.0007x_3 + 0.05x_4 - 0.0025x_5 - 0.0024x_7 + 0.0009x_9 + 0.05u_R + 0.022u_T + 0.01u_C$$

$$\dot{x}_3 = -0.00011x_1 + 0.000036x_2 - 0.3242x_3 + 0.0000057x_5 + 0.00018x_7 - 0.424u_C$$

$$\dot{x}_4 = x_5$$

$$\dot{x}_5 = 2.61x_1 - 7.25x_2 - 0.163x_3 - 1.96x_5 - 1.94x_7 + 0.01x_9 + 21.81u_R + 0.3475u_T - 2.13u_C$$

$$\dot{x}_6 = x_7$$

$$\dot{x}_7 = 1.97x_1 + 0.736x_2 + 1.0x_3 + 0.548x_5 - 0.542x_7 - 6.15u_P + 0.69u_C$$

$$\dot{x}_8 = x_9$$

$$\dot{x}_9 = 0.016x_1 + 5.59x_2 - 0.193x_3 - 0.0043x_5 - 0.0083x_7 - 0.303x_9 + 0.174u_P - 7.48u_T + 5.12u_C$$

where the state variables $x_1, x_2 \ldots x_9$ are

$x_1, x_2, x_3 = \mu_X, \mu_Y, \mu_Z$ = longitudinal, lateral, and vertical velocities (normalized)

$x_4, x_6, x_8 = \phi, \theta_F, \psi$, roll, pitch, and yaw angles (radians)

$x_5, x_7, x_9 = \dot{\phi}, \dot{\theta}_F, \dot{\psi}$, roll, pitch, and yaw rates (radians)

and the control functions u_P, u_R, u_T, u_C are

u_P = longitudinal cyclic pitch control input (radians)

u_R = lateral cyclic pitch control input (radians)

u_T = tail rotor collective pitch control input (radians)

u_C = main rotor collective pitch control input (radians)

Using the state vector notation, the equations may be represented in the form $\dot{x} = Ax + Bu$, where

$$A = \begin{bmatrix} -0.016 & -0.0047 & 0 & 0 & -0.0001 & -0.05 & 0.0025 & 0 & 0 \\ 0.0047 & -0.033 & -0.0007 & 0.05 & -0.0025 & 0 & -0.0024 & 0 & 0.0009 \\ -0.000011 & 0.000036 & -0.3242 & 0 & 0.0000057 & 0 & 0.00018 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2.61 & -7.25 & -0.163 & 0 & -1.96 & 0 & -1.94 & 0 & 0.01 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1.97 & 0.736 & 1.0 & 0 & 0.548 & 0 & -0.542 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0.016 & 5.59 & -0.193 & 0 & -0.0043 & 0 & -0.0083 & 0 & -0.303 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.05 & 0 & 0 & 0.005 \\ 0 & 0.05 & 0.022 & 0.01 \\ 0 & 0 & 0 & -0.424 \\ 0 & 0 & 0 & 0 \\ 0 & 21.81 & 0.3475 & -2.13 \\ 0 & 0 & 0 & 0 \\ -6.15 & 0 & 0 & 0.69 \\ 0 & 0.174 & -7.48 & 5.12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Performance Index

a) Regulator problem

The performance index for the regulator problem is selected to be

$$2J = \int_0^\infty (\alpha_{11}x_1^2 + \alpha_{22}x_2^2 + \alpha_{33}x_3^2 + \alpha_{88}x_8^2 + \beta_{11}u_P^2 + \beta_{22}u_R^2 + \beta_{33}u_T^2 + \beta_{44}u_C^2)dt$$

Since

$$2J = x(T)'Fx(T) + \int_0^\infty (x'Qx + u'Ru)dt$$

the matrices Q, R, and F are given by

$$Q = \begin{bmatrix} \alpha_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{88} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} \beta_{11} & 0 & 0 & 0 \\ 0 & \beta_{22} & 0 & 0 \\ 0 & 0 & \beta_{33} & 0 \\ 0 & 0 & 0 & \beta_{44} \end{bmatrix} \qquad F = [0] \text{ the null matrix}$$

If $\alpha_{ii} \geq 0 (i = 1,2,3,8)$ and $\beta_{jj} > 0$ $(j = 1,2,3,4)$, the matrix Q is positive semidefinite, and the matrix R is positive definite. Hence the techniques of Sec. 2.0 apply, and the regulator problem can be solved on a digital computer by assigning values to the α and β elements. Only four diagonal elements in the Q matrix have nonzero values. These terms penalize velocity (longitudinal, lateral, and vertical) and heading deviations from trim conditions. If the steady-state values of these variables are zero, the steady-state values of the remaining state variables will also be zero.

General comments have been made in Sec. 4.0 concerning the updating of the Q and R matrices. In a hovering helicopter, the response of each channel is primarily due to one control input (e.g., rate of climb due to collective stick input). This characteristic, which is common to most aircraft control systems, is used to determine a systematic method of adjusting α_{ii} in the Q matrix and the corresponding value of β_{jj} in the R matrix. Since the amplitude of the disturbance experienced by the helicopter, the speed of response desired by the pilot, and the amplitude constraint on the controller output are important parameters in the design, the adjustment of α_{ii} and β_{jj} is carried out with reference to the values of these parameters.

The various steps in the determination of a suitable Q matrix with $R = [I]$ may be outlined as follows:

1) For an initial choice of the elements α_{ii} $(i = 1,2,3,8)$ of the Q matrix, the Riccati equation is solved to yield a set of feedback equations.

2) The system response is determined for an initial condition x_{10}^* on the state x_1, and zero initial conditions on the remaining state variables. The controller output in channel $1(u_P)$ is scaled so that its peak excursion coincides with the maximum allowable controller output. This is accomplished by scaling the initial condition x_{10}^* to x_{10} since the system is linear. The system speed of response is measured as the time for $x_1(t)$, the velocity, to reach zero [i.e., $x_1(t_{10}) = 0$].

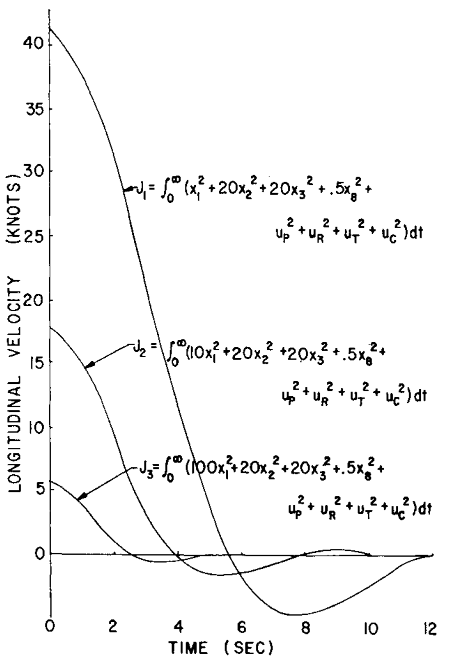


Fig. 3 Response of longitudinal velocity to a disturbance with α_{11} = 1, 10, and 100.

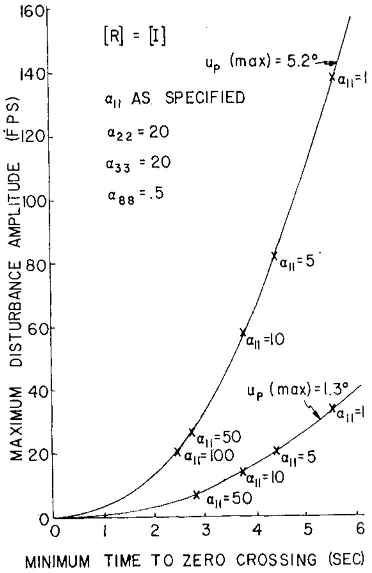


Fig. 4 Maximum disturbance amplitude vs minimum longitudinal velocity response time with 5 and 20% controller constraints.

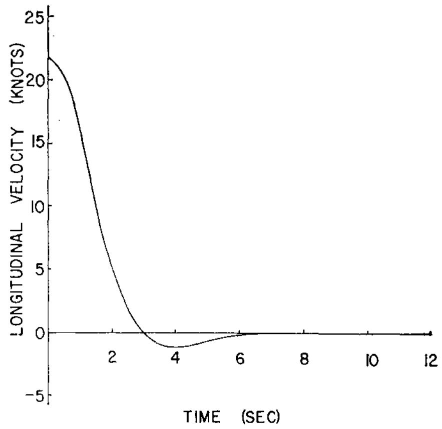
3) x_{10} and t_{10} represent the maximum amplitude of disturbance that can be tolerated in the longitudinal channel, and the corresponding minimum time for zero crossing which can be realized, without exceeding the amplitude constraint on the controller output u_P. In Fig. 2, a curve C_1 relating x_{10} and t_{10} is shown with α_{11} as a parameter. The experiment described in step 2 yields a single point on this curve. By selecting various values of α_{11}, the curve C_1 is drawn. Figure 3 indicates that as α_{11} is increased, x_{10} and t_{10} decrease monotonically but the form of the response curve remains unaltered. For a specified value of α_{11}, wide variations in the magnitudes of α_{ii} $(i \neq 1)$ are found to have no appreciable effect on the corresponding values of x_{10} and t_{10} (Fig. 2). It is this fact that enables the elements of the Q matrix to be determined independently. For a specified magnitude of disturbance x_{10}, a faster response can be achieved only by increasing the amplitude constraint on the controller output. This is shown in Fig. 4, where two curves are shown corresponding to two values of the amplitude constraint.

4) Curves C_2, C_3, and C_4 relating α_{ii}, the maximum amplitude of disturbance x_{i0}, and the corresponding speed of response t_{i0} in the other channels can be obtained in a similar fashion.

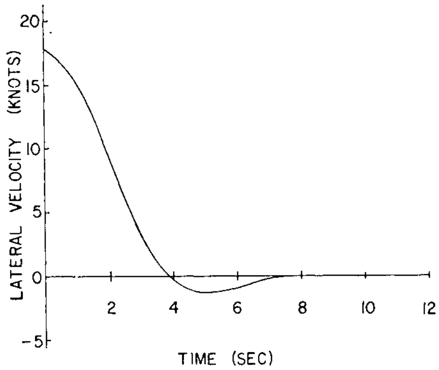
5) The curves C_1 to C_4 form the basis for the selection of the feedback gains. For specified maximum disturbance and controller amplitude constraint in each channel, the elements α_{ii} of the Q matrix are determined from the curves. The solution of the Riccati equation using this Q matrix (and $R = I$) yields the desired feedback gains.

The fact that the response in a particular channel is unaffected by variations in the penalty weightings on other channels suggests that the resultant optimal system is decoupled. Examination of the closed-loop state transition matrix $[A - BR^{-1}B'K]$ reveals that this is indeed the case. By penalizing velocity and heading deviations about the trimmed value, the designer is, in essence, forcing the closed-loop optimal system to eliminate response interaction between these states.

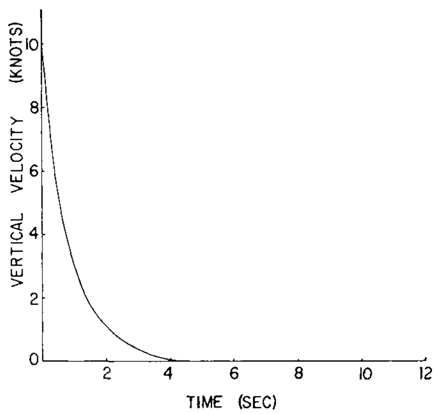
The results of applying the preceding technique to the SH-3D are shown in Fig. 5. This particular system was chosen from the group of optimal systems corresponding to the curve of Fig. 2 and from similar curves for the remaining three channels. The values of $\alpha(\alpha_{11} = \alpha_{22} = \alpha_{33} = 10$, $\alpha_{88} = 0.5)$ were obtained from the curves, assuming $\beta_{jj} = 1(j = 1, \ldots 4)$ for the amplitude of the disturbances expected in each channel, and the response times desired by the pilot. Using these values in the Q and R matrices, the optimal feedback gains [Eq. (15)] and the corresponding outputs of the



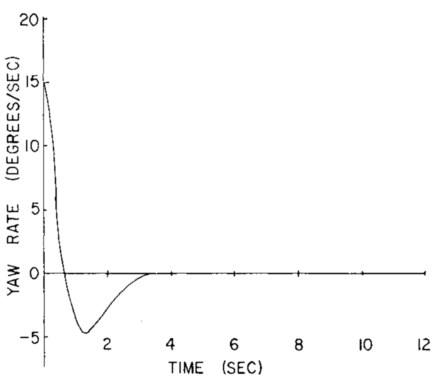
a) Longitudinal velocity



b) Lateral velocity

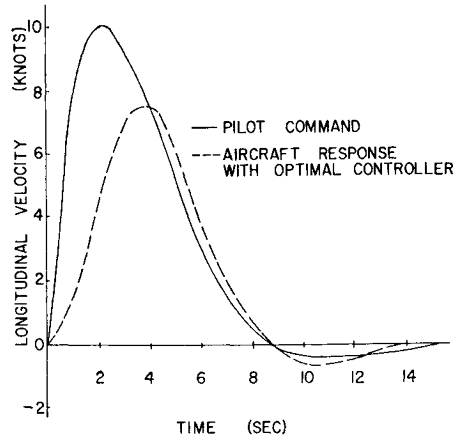


c) Vertical velocity

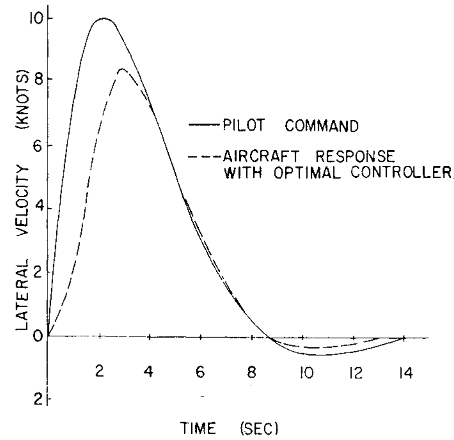


d) Yaw rate

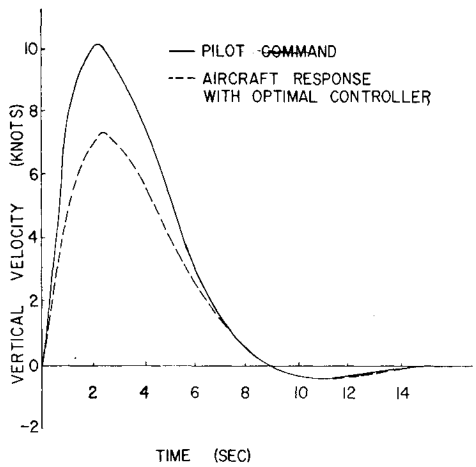
Fig. 5 Final system regulator: response to a disturbance; $J = \int_0^\infty (10\,x_1^2 + 10\,x_2^2 + 10\,x_8^2 + 0.5\,x_8^2 + U_P^2 + U_R^2 + U_T^2 + U_C^2)\,dt$



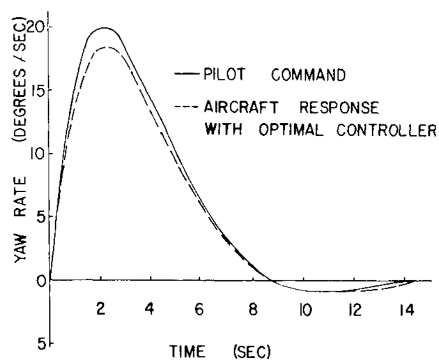
a) Longitudinal velocity



b) Lateral velocity



c) Vertical velocity



d) Yaw rate

Fig. 6 Aircraft response to a pilot pulse command with optimal controller.

system (Fig. 5) were calculated:

$$-R^{-1}B'K = \begin{bmatrix} 2.59 & 1.19 & -0.12 & 0.10 & 0 & -0.28 & -0.20 & 0.01 & 0 \\ -1.16 & 2.66 & 0.04 & 0.20 & 0.07 & 0.11 & -0.01 & 0.02 & 0.01 \\ -0.05 & -0.58 & -0.99 & -0.01 & 0 & 0.01 & 0 & -0.22 & -0.28 \\ -0.02 & 0.14 & -2.15 & -0.01 & 0 & 0 & 0 & 0.09 & 0.15 \end{bmatrix} \quad (15)$$

b) Model following

The regulator problem of the preceding section is easily extended to encompass the control problem as shown in Sec. 3.0. In this case, it is necessary to determine the feedforward gains assuming that the general form of the pilot's command input is known.

Pilot commands, introduced into the system as shown in Fig. 1, are generated as outputs of a model with certain initial conditions. For example, typical pilot inputs in a helicopter are pulses of variable duration, amplitude, and slope. These inputs can be described by the solution of a homogeneous second-order differential equation of the form

$$d^2y/dt^2 + a_1(dy/dt) + a_0y = 0$$

or

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad (16)$$

with an initial condition $\dot{y}(0)$. By varying a_0, a_1 and $\dot{y}(0)$, a wide range of command signals can be generated. For a specific a_0 and a_1, changing $\dot{y}(0)$ merely changes the amplitude of the response. The equations of the model (16) are adjoined to the equations of the aircraft, and the entire system is solved as a regulator problem using the technique of Sec. 3.0.

The response of the SH-3D to various pulse inputs is shown in Fig. 6. The optimal controller for each input was determined using the procedure discussed previously. The feedforward gains are found to depend on the model, whereas the feedback gains are the same as those determined in Sec. 5a. Thus, as previously discussed, a fixed feedforward controller can be optimal only for one specific input. However, it is possible to choose a fixed gain suboptimal controller that yields satisfactory response over the range of inputs being considered.

The fixed feedforward gains were determined by solving the optimal control problem for the most common pilot inputs. The response of the suboptimal system to other inputs is found to be quite satisfactory (Fig. 7).

Since the performance index is a quadratic function in the error e and the control function u, a steady-state error is incurred in following step inputs (Fig. 7). If the control function is not included in the performance index, the steady-state value of e will be zero. Similarly, if only the control function is included the feedforward and feedback gains will be identically zero since $Q, Q_1 = 0$.

In general, the final solution is a compromise and depends upon the steady-state error and control function that can be tolerated. Although, for individual channels, scaling can be used to make the steady-state value of the output equal the input, such a method cannot be adopted for a coupled multivariable system subject to many simultaneous inputs.

6.0 Conclusions

Optimal control theory has provided a convenient method of determining the feedback gains of a linear, time-invariant

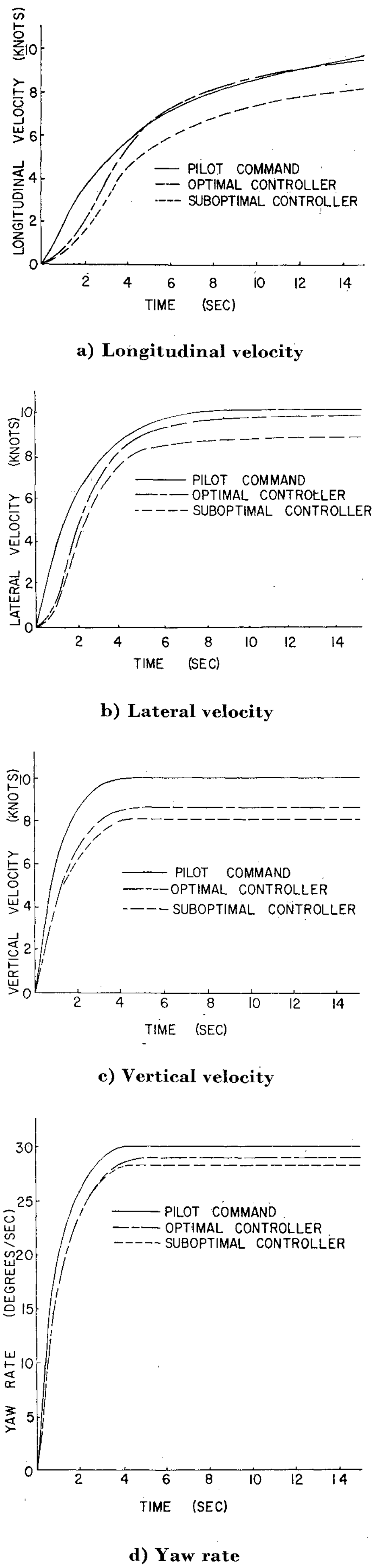


Fig. 7 Comparison between optimal and suboptimal controllers.

system which minimize a quadratic performance index. Application of these techniques to the design of helicopter stability augmentation systems has revealed that certain response characteristics can be related to the parameters in the performance index. A relationship between the specified speed of response of the closed-loop optimal system and the maximum amplitude of a disturbance is determined for a specified amplitude constraint on the controller. The particular performance index which corresponds to these specifications yields the required feedback gains. The example in Sec. 5a indicates that this procedure results in a decoupling of the major response modes of the aircraft so that performance for each mode can be improved separately.

The choice of a performance index, rather than feedback gains, to obtain the desired response is the central feature of the method. The solutions to various optimal control problems merely constitute intermediate steps in the search for a response that will be satisfactory to the pilot. The control which the designer exercises over the response while adjusting the various parameters of the performance index, and the availability of an analytical method for determining the response for a given performance index, make the approach both attractive and feasible.

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